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Derivation of functional equations for Feynman integrals from algebraic relations

O.V. Tarasov

II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany

and

Joint Institute for Nuclear Research,
141980 Dubna, Russian Federation

E-mail: otarasov@jinr.ru

Abstract

New methods for obtaining functional equations for Feynman integrals are presented. Application of these methods for finding functional equations for various one- and two-loop integrals described in detail. It is shown that with the aid of functional equations Feynman integrals in general kinematics can be expressed in terms of simpler integrals.

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1 Introduction

Recently it was discovered that Feynman integrals obey functional equations [1], [2]. Different examples of functional equations were presented in Refs. [1], [3],[2]. In these articles only one-loop integrals were considered.

In the present paper we propose essentially new methods for deriving functional equations. These methods are based on algebraic relations between propagators and they are suitable for deriving functional equations for multi-loop integrals. Also these methods can be used to derive functional equations for integrals with some propagators raised to non-integer powers.

Our paper is organized as follows. In Sec. 2. the method proposed in Ref. [1] is shortly reviewed.

In Sec. 3. a method for finding algebraic relations between products of propagators is formulated. We describe in detail derivation of explicit relations for products of two, three and four propagators. Also algebraic relation for products of arbitrary number of propagators is given. These relations are used in Sec.4. to obtain functional equations for some one-, as well as two- loop integrals. In particular functional equation for the massless one-loop vertex type integral is presented. Also functional equation for the two-loop vertex type integral with arbitrary masses is given.

In Sec. 5. another method for obtaining functional equations is proposed. The method is based on finding algebraic relations for ‘deformed propagators’ and further conversion of integrals with ‘deformed propagators’ to usual Feynman integrals by imposing conditions on deformation parameters. To perform such a conversion the α - parametric representation for

both types of integrals is exploited. The method was used to derive functional equation for the two-loop vacuum type integral with arbitrary masses. As a by product, from this functional equation we obtained new hypergeometric representation for the one-loop massless vertex integral.

In conclusion we formulate our vision of the future applications and developments of the proposed methods.

2 Deriving functional equations from recurrence relations

The method for deriving functional equations proposed in Ref. [1] is based on the use different kind of recurrence relations. In particular in Refs. [1], [2], [3], generalized recurrence relations [4] were utilized to obtain functional equations for one-loop Feynman integrals. In general such recurrence relations connect a combination of some number of integrals $I_{1,n}, \dots, I_{k,n}$ corresponding to diagrams, say, with n lines and integrals corresponding to diagrams with fewer number of lines. Diagrams with fewer number of lines can be obtained by contracting some lines in integrals with n lines. Integrals corresponding to such diagrams depend on fewer number of kinematical variables and masses compared to integrals with n lines. Such recurrence relations can be written in the following form:

$$\sum_j Q_j(\{m_i\}, \{s_q\}, \nu_l, d) I_{j,n} = \sum_{k,r < n} R_{k,r}(\{m_i\}, \{s_m\}, \nu_l, d) I_{k,r}, \quad (2.1)$$

where Q_j and R_k are ratios of polynomials depending on masses m_i , scalar products s_r of external momenta, powers of propagators ν_l and parameter of the space time dimension d . At the left hand-side of Eq. (2.1) we combined integrals with n lines and on the right hand - side integrals with fewer number of lines.

In accordance with the method of Ref. [1], to obtain functional equation from Eq. (2.1) one should eliminate terms on the left hand - side by defining some kinematical variables from the set of equations:

$$Q_j(\{m_i\}, \{s_q\}, \nu_l, d) = 0. \quad (2.2)$$

If there is a nontrivial solution of this system and for this solution some $R_{k,r}(\{m_i\}, \{s_m\}, \nu_l, d)$ are different from zero then the right-hand side of Eq. (2.1) will represent functional equation.

For the one-loop integrals with n propagators

$$I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}) = \int \frac{d^d q}{i\pi^{d/2}} \prod_{j=1}^n \frac{1}{P_j^{\nu_j}}, \quad (2.3)$$

where

$$P_j = (q - p_j)^2 - m_j^2 + i\epsilon, \quad (2.4)$$

different types of recurrence relations were given in Refs. [4], [5]. Diagram corresponding to this integral is given in Figure 1. In Refs. [4], [5] the following relation was derived:

$$\begin{aligned} & (d - \sum_{i=1}^n \nu_i + 1) G_{n-1} I_n^{(d+2)}(\{m_j^2\}, \{s_{kl}\}) - 2\Delta_n I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}) \\ & = \sum_{k=1}^n (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}), \end{aligned} \quad (2.5)$$

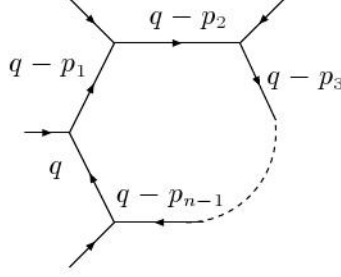


Figure 1: One-loop diagram with n external legs

where the operators \mathbf{k}^- shift index of propagators by one unit $\nu_k \rightarrow \nu_k - 1$,

$$G_{n-1} = -2^n \begin{vmatrix} (p_1 - p_n)(p_1 - p_n) & (p_1 - p_n)(p_2 - p_n) & \dots & (p_1 - p_n)(p_{n-1} - p_n) \\ (p_1 - p_n)(p_2 - p_n) & (p_2 - p_n)(p_2 - p_n) & \dots & (p_2 - p_n)(p_{n-1} - p_n) \\ \vdots & \vdots & \ddots & \vdots \\ (p_1 - p_n)(p_{n-1} - p_n) & (p_2 - p_n)(p_{n-1} - p_n) & \dots & (p_{n-1} - p_n)(p_{n-1} - p_n) \end{vmatrix}, \quad (2.6)$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix},$$

$$Y_{ij} = m_i^2 + m_j^2 - s_{ij}, \quad s_{ij} = (p_i - p_j)^2. \quad (2.7)$$

Here p_i, p_j are external momenta going through lines i, j respectively, and m_j is mass attributed to j -th line. Gram determinant G_{n-1} and modified Cayley determinant Δ_n are polynomials depending on scalar products and masses.

It is assumed that these scalar products are made of d dimensional vectors and G_{n-1} and Δ_n are not subject to any restriction or condition specific to some integer values of d . Eq. (2.5) is written in the form corresponding to Eq. (2.1). To eliminate integrals with n lines on the left hand - side of Eq. (2.5) the following conditions to be hold:

$$G_{n-1} = 0, \quad \Delta_n = 0. \quad (2.8)$$

Eq. (2.5) is valid for arbitrary kinematical variables and masses. Solution of Eqs. (2.8) can be easily done with respect to two kinematical variables or masses. Starting from $n = 3$ substitution of such solutions into Eq. (2.5) gives nontrivial functional equations.

The method for obtaining functional equations by eliminating complicated integrals from recurrence relations is quite general one. However for multi loop integrals, depending on several kinematical variables, derivation of equations like Eq. (2.5) is computationally challenging. In the next sections we will describe easier and more powerful methods that can be used for deriving functional equations for multi-loop integrals.

3 Deriving functional equations from algebraic relations between propagators

Setting $\nu_j = 1$ in Eq. (2.5) and imposing conditions (2.8) leads to the following equation:

$$\sum_{k=1}^n (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)} = 0. \quad (3.9)$$

In Eq. (3.9) integrands of $\mathbf{k}^- I_n^{(d)}$ are products of $n - 1$ propagators depending on different external momenta, i.e. each term in this relation corresponds to the same function but with different arguments. In fact functional equations considered in Refs. [1, 3, 2] are of the same form as Eq. (3.9). The question naturally arises: This relationship holds for integrals or it can be obtained as the consequence of a relationship between integrands?

By inspecting Eq. (3.9), one can suggest the following form of the relation between products of propagators of integrands:

$$\prod_{r=1}^n \frac{1}{P_r} = \frac{1}{P_{n+1}} \sum_{r=1}^n x_r \prod_{\substack{j=1 \\ j \neq r}}^n \left(\frac{1}{P_j} \right), \quad (3.10)$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\epsilon. \quad (3.11)$$

In what follows we will omit $i\epsilon$ term assuming that all masses have such a correction. Additionally we assume that vectors p_j are linearly dependent, i.e. the Gram determinant for the set of vectors $\{p_j\}$ is equal to zero. Such a condition is valid for all examples considered in Refs. [1], [2].

Now let's consider in detail implementation of our prescription for products of 2,3 and 4 propagators. At $n = 2$ relation (3.10) reads:

$$\frac{1}{P_1 P_2} = \frac{x_1}{P_2 P_3} + \frac{x_2}{P_1 P_3}, \quad (3.12)$$

where

$$P_1 = (k_1 - p_1)^2 - m_1^2, \quad P_2 = (k_1 - p_2)^2 - m_2^2, \quad P_3 = (k_1 - p_3)^2 - m_3^2. \quad (3.13)$$

According to our assumption three vectors p_1, p_2, p_3 are linearly dependent. Without loss of generality we may assume that

$$p_3 = y_{31} p_1 + y_{32} p_2. \quad (3.14)$$

Furthermore, we assume that k_1 will be integration momentum and scalar quantities x_1, x_2, y_{32}, y_{31} do not depend on k_1 . Putting all terms in Eq. (3.12) over a common denominator and then equating to zero the coefficients in front of various products of $k_1^2, k_1 p_1, k_1 p_2$ yields the following system of equations:

$$\begin{aligned} y_{32} - x_2 &= 0, & y_{31} - x_1 &= 0, & x_1 + x_2 &= 1, \\ p_1^2(x_1 - y_{31}^2) + p_2^2(x_2 - y_{32}^2) + y_{31}y_{32}(s_{12} - p_1^2 - p_2^2) - m_1^2 x_1 - m_2^2 x_2 + m_3^2 &= 0. \end{aligned} \quad (3.15)$$

Solution of this system of equations is:

$$x_1 = y_{31} = \lambda_2, \quad x_2 = y_{32} = 1 - \lambda_2, \quad (3.16)$$

where λ_2 is a root of the equation

$$A_2\lambda_2^2 + B_2\lambda_2 + C_2 = 0, \quad (3.17)$$

with

$$A_2 = s_{12}, \quad B_2 = m_1^2 - m_2^2 - s_{12}, \quad C_2 = m_2^2 - m_3^2. \quad (3.18)$$

This solution can be rewritten in an explicit form:

$$\begin{aligned} x_1 = y_{31} &= \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{\Lambda_2 + 4s_{12}m_3^2}}{2s_{12}}, \\ x_2 = y_{32} &= \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} \mp \frac{\sqrt{\Lambda_2 + 4s_{12}m_3^2}}{2s_{12}}, \end{aligned} \quad (3.19)$$

where

$$\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}m_1^2 - 2s_{12}m_2^2 - 2m_1^2m_2^2. \quad (3.20)$$

Now let's find algebraic relation for the products of three propagators. At $n = 3$ Eq. (3.10) reads:

$$\frac{1}{P_1P_2P_3} = \frac{x_1}{P_4P_2P_3} + \frac{x_2}{P_1P_4P_3} + \frac{x_3}{P_1P_2P_4}, \quad (3.21)$$

where P_1, P_2, P_3 are defined in Eq.(3.13) and

$$P_4 = (k_1 - p_4)^2 - m_4^2. \quad (3.22)$$

In complete analogy with the previous case we can represent one momentum as a combination of other ones. Without loss of generality we may write

$$p_4 = y_{41}p_1 + y_{42}p_2 + y_{43}p_3, \quad (3.23)$$

where y_{ij} for the time being are arbitrary coefficients. Putting all terms in Eq. (3.21) over a common denominator and then equating to zero the coefficients in front of various products of $(k_1^2), (k_1p_1), (k_1p_2), (k_1p_3)$ yields the following system of equations:

$$\begin{aligned} y_{43} - x_3 &= 0, \quad y_{42} - x_2 = 0, \quad y_{41} - x_1 = 0, \quad x_1 + x_2 + x_3 = 1, \\ p_1^2(x_1 - y_{41}^2) &+ p_2^2(x_2 - y_{42}^2) + p_3^2(x_3 - y_{43}^2) \\ &+ y_{42}y_{43}(s_{23} - p_2^2 - p_3^2) + y_{41}y_{43}(s_{13} - p_1^2 - p_3^2) + y_{41}y_{42}(s_{12} - p_1^2 - p_2^2) \\ &- m_1^2x_1 - m_2^2x_2 - m_3^2x_3 + m_4^2 = 0. \end{aligned} \quad (3.24)$$

Solving these equations for $x_1, x_2, x_3, y_{41}, y_{42}$ we have

$$x_1 = 1 - \lambda_3 - y_{43}, \quad x_2 = \lambda_3, \quad x_3 = y_{43}, \quad y_{41} = 1 - y_{43} - \lambda_3, \quad y_{42} = \lambda_3, \quad (3.25)$$

where λ_3 is solution of the equation

$$A_3\lambda_3^2 + B_3\lambda_3 + C_3 = 0. \quad (3.26)$$

Here

$$\begin{aligned} A_3 &= s_{12}, \\ B_3 &= y_{43}(s_{13} + s_{12} - s_{23}) - m_1^2 + m_2^2 - s_{12}, \\ C_3 &= y_{43}^2s_{13} + (m_3^2 - m_1^2 - s_{13})y_{43} + m_1^2 - m_4^2. \end{aligned} \quad (3.27)$$

Let us now turn to the derivation of algebraic relation for the product of four propagators. At $n = 4$ Eq. (3.10) reads:

$$\frac{1}{P_1 P_2 P_3 P_4} = \frac{x_1}{P_5 P_2 P_3 P_4} + \frac{x_2}{P_1 P_5 P_3 P_4} + \frac{x_3}{P_1 P_2 P_5 P_4} + \frac{x_4}{P_1 P_2 P_3 P_5}, \quad (3.28)$$

where P_1, P_2, P_3, P_4 are defined in Eqs. (3.13), (3.22),

$$P_5 = (k_1 - p_5)^2 - m_5^2, \quad (3.29)$$

and p_5 is a linear combination of vectors p_1, \dots, p_4 ,

$$p_5 = y_{51}p_1 + y_{52}p_2 + y_{53}p_3 + y_{54}p_4. \quad (3.30)$$

Putting all terms in Eq. (3.28) over a common denominator and then equating to zero the coefficients in front of different products of $k_1^2, k_1 p_j$ yields system of equations:

$$\begin{aligned} y_{51} - x_1 &= 0, & y_{52} - x_2 &= 0, & y_{53} - x_3 &= 0, & y_{54} - x_4 &= 0, \\ x_1 + x_2 + x_3 + x_4 &= 1, \\ m_5^2 - m_1^2 x_1 - m_2^2 x_2 - m_3^2 x_3 - m_4^2 x_4 + p_4^2 (x_4 - y_{54}^2) \\ &+ p_3^2 (x_3 - y_{53}^2) + p_2^2 (x_2 - y_{52}^2) + p_1^2 (x_1 - y_{51}^2) + (s_{12} - p_1^2 - p_2^2) y_{51} y_{52} \\ &+ (s_{13} - p_1^2 - p_3^2) y_{51} y_{53} + (s_{23} - p_2^2 - p_3^2) y_{52} y_{53} + (s_{14} - p_1^2 - p_4^2) y_{54} y_{51} \\ &+ (s_{24} - p_2^2 - p_4^2) y_{54} y_{52} + (s_{34} - p_3^2 - p_4^2) y_{54} y_{53} = 0. \end{aligned} \quad (3.31)$$

Solving this system for $x_1, x_2, x_3, x_4, y_{51}, y_{54}$ we have

$$\begin{aligned} x_1 &= y_{51} = \lambda_4, & x_2 &= y_{52}, & x_3 &= y_{53}, \\ x_4 &= y_{54} = 1 - x_1 - x_2 - x_3 = 1 - y_{53} - y_{52} - \lambda_4, \end{aligned} \quad (3.32)$$

where λ_4 is a solution of the equation

$$A_4 \lambda_4^2 + B_4 \lambda_4 + C_4 = 0, \quad (3.33)$$

with

$$\begin{aligned} A_4 &= s_{14}, \\ B_4 &= (s_{24} - s_{12} + s_{14}) y_{52} + (s_{34} - s_{13} + s_{14}) y_{53} + m_1^2 - m_4^2 - s_{14}, \\ C_4 &= s_{24} y_{52}^2 + (s_{34} - s_{23} + s_{24}) y_{52} y_{53} + (m_2^2 - m_4^2 - s_{24}) y_{52} + s_{34} y_{53}^2 \\ &+ (m_3^2 - m_4^2 - s_{34}) y_{53} + m_4^2 - m_5^2. \end{aligned} \quad (3.34)$$

Eqs. (3.21), (3.21) and (3.28) will be used in the next sections to derive functional equations for the propagator, vertex and box type of integrals. Relations between products of five and more propagators can be easily derived in the same way as it was done for products of two-, three- and four- propagators. From Eq. (3.10) one can derive system of equations and find its solution for arbitrary n . Multiplying both sides of Eq. (3.10) by the product of $n + 1$ propagators $\prod_{j=1}^{n+1} P_j$ yields

$$P_{n+1} = \sum_{r=1}^n P_r, \quad (3.35)$$

or

$$k_1^2 - 2k_1 p_{n+1} + p_{n+1}^2 = \sum_{r=1}^n x_r (k_1^2 - 2k_1 p_r + p_r^2 - m_r^2). \quad (3.36)$$

Since we assume linear dependence of vectors p_r , without loss of generality we may write:

$$p_{n+1} = \sum_{j=1}^n y_{n+1,j} p_j. \quad (3.37)$$

Substituting (3.37) into Eq.(3.36), collecting terms in front of k_1^2 , $k_1 p_j$ and terms without k_1 , equating them to zero after some simplifications yields the following system of $n + 2$ equations:

$$x_k - y_{n+1,k} = 0, \quad k = 1, \dots, n, \quad (3.38)$$

$$\sum_{k=1}^n y_{n+1,k} = 1, \quad (3.39)$$

$$m_{n+1}^2 - \sum_{k=1}^n y_{n+1,k} m_k^2 + \sum_{j=1}^n \sum_{l=1}^{j-1} y_{n+1,j} y_{n+1,l} s_{lj} = 0. \quad (3.40)$$

Solving Eq. (3.39) for one of the y_{ij} and substituting this solution into Eq. (3.40) gives quadratic equation for the remaining y_{ij} . This quadratic equation can be solved with respect to one of the parameters y_{ij} . Thus the solution of the system of equations (3.38), (3.39), (3.40) will depend on $n - 2$ arbitrary parameters y_{ij} and one arbitrary mass m_n .

It is interesting to note that for any n , functional equations for integrals with all masses equal to zero and functional equations for integrals with all masses equal are the same. In case of equal masses, two mass dependent terms in Eq. (3.40) cancel each other due to Eq. (3.39). In both cases systems of equations for x_i , y_{jk} are the same and therefore arguments of integrals are the same.

Eq. (3.12) is analogous to the equation for splitting propagators presented in Ref. [6]. Eq. (3.21) is a generalization of Eq. (3.12). Indeed, setting $y_{43} = x_3 = 0$, canceling common factor P_3 on both sides of Eq. (3.21) yields relation similar to (3.12). In turn Eq. (3.28) is a generalization of (3.21).

3.1 Prototypes of functional equations

Multiplying algebraic relations (3.12), (3.21), (3.28) by products of any number of propagators raised to arbitrary powers ν_j

$$\prod_{j=n_0}^N \frac{1}{[(k_1 - p_j) - m_j^2]^{\nu_j}} \quad (3.41)$$

and integrating with respect to k_1 we get a functional equation for one-loop integrals. Eqs. (3.12), (3.21), (3.28) also can be used to derive functional equations for integrals with any number of loops. Multiplying algebraic relations for propagators by function corresponding to Feynman integral depending on momentum k_1 and any number of external momenta and then integrating with respect to k_1 will produce functional equations. Just for demonstrational purposes we present graphically in Figure 2 functional equation based on n propagator relation. The blob on this picture correspond to either product of propagators raised to arbitrary powers or to an integral with any number of loops and external legs. One of the external momenta of this multi loop integral should be k_1 .

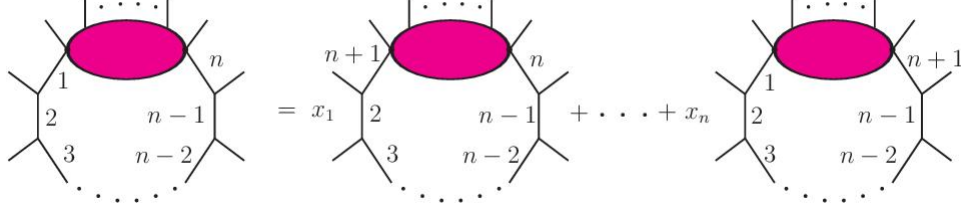


Figure 2: n - propagator functional equation

4 Some examples of functional equations

In this section several particular examples of functional equations resulting from algebraic relations for products of propagators will be considered.

4.1 Functional equation for the one-loop propagator type integral

First, we consider the simplest case, namely, functional equation for the integral $I_2^{(d)}$:

$$I_2^{(d)}(m_j^2, m_k^2; s_{jk}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1 - p_j)^2 - m_j^2][(k_1 - p_k)^2 - m_k^2]}. \quad (4.42)$$

Integrating both sides of Eq. (3.12) with respect to k_1 , we get:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_3^2, s_{23}) + x_2 I_2^{(d)}(m_1^2, m_3^2, s_{13}). \quad (4.43)$$

The arguments s_{13} , s_{23} of integrals on the right hand - side depend on y_{31} , y_{32}

$$\begin{aligned} s_{13} &= (p_1 - p_3)^2 = (y_{31} - 1)^2 p_1^2 + 2y_{32}(y_{31} - 1)p_1 p_2 + y_{32}^2 p_2^2, \\ s_{23} &= (p_2 - p_3)^2 = p_1^2 y_{31}^2 + 2(y_{32} - 1)y_{31} p_1 p_2 + (y_{32} - 1)^2 p_2^2. \end{aligned} \quad (4.44)$$

Substituting solution for y_{ij} from Eq. (3.16) into Eq. (4.44) yields:

$$\begin{aligned} s_{13} &= \frac{m_2^4 - 2m_2^2 m_1^2 - 2m_2^2 s_{12} + m_1^4 + s_{12}^2 + 2s_{12} m_3^2}{2s_{12}} \\ &\quad \pm \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} \sqrt{\Lambda_2 + 4s_{12} m_3^2} \\ s_{23} &= \frac{s_{12}^2 - 2m_1^2 s_{12} + m_1^4 + m_2^4 - 2m_1^2 m_2^2 + 2s_{12} m_3^2}{2s_{12}} \\ &\quad \pm \frac{m_1^2 - m_2^2 - s_{12}}{2s_{12}} \sqrt{\Lambda_2 + 4s_{12} m_3^2}. \end{aligned} \quad (4.45)$$

In this equation m_3^2 is an arbitrary parameter and can be taken at will. Functional equation (4.43) is in agreement with the result presented in Refs. [1],[2].

4.2 Functional equations for the one-loop vertex type integral

Functional equations for the vertex type integral

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1 - p_1)^2 - m_1^2][(k_1 - p_2)^2 - m_2^2][(k_1 - p_3)^2 - m_3^2]}, \quad (4.46)$$

can be obtained from Eq. (3.12) as well as from Eq. (3.21). Multiplying Eq. (3.12) with the factor $1/P_4$ where

$$P_4 = (k_1 - p_4)^2 - m_4^2, \quad (4.47)$$

and integrating over k_1 leads to the equation:

$$\int \frac{d^d k_1}{P_1 P_2 P_4} = x_1 \int \frac{d^d k_1}{P_2 P_3 P_4} + x_2 \int \frac{d^d k_1}{P_1 P_3 P_4}. \quad (4.48)$$

This equation in terms of integrals $I_3^{(d)}$ reads

$$I_3^{(d)}(m_1^2, m_2^2, m_4^2; s_{24}, s_{14}, s_{12}) = x_1 I_3^{(d)}(m_2^2, m_3^2, m_4^2; s_{34}, s_{24}, s_{23}) + x_2 I_3^{(d)}(m_1^2, m_3^2, m_4^2; s_{34}, s_{14}, s_{13}). \quad (4.49)$$

Two more functional equations can be obtained from Eq. (4.48) by symmetric permutations $(m_1, p_1) \leftrightarrow (m_4, p_4)$ and $(m_2, p_2) \leftrightarrow (m_4, p_4)$.

Another functional equation for the vertex type integral can be obtained by integrating Eq. (3.21) with respect to k_1 :

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = (1 - \lambda_3 - y_{43}) I_3^{(d)}(m_4^2, m_2^2, m_3^2, s_{23}, s_{34}, s_{24}) + \lambda_3 I_3^{(d)}(m_1^2, m_4^2, m_3^2, s_{34}, s_{13}, s_{14}) + y_{43} I_3^{(d)}(m_1^2, m_2^2, m_4^2, s_{24}, s_{14}, s_{13}), \quad (4.50)$$

where

$$\begin{aligned} s_{14} &= (m_1^2 - m_2^2 + s_{12})\lambda_3 + (m_1^2 - m_3^2 + s_{13})y_{43} + m_4^2 - m_1^2, \\ s_{24} &= (m_1^2 - s_{12} - m_2^2)\lambda_3 + (m_1^2 - m_3^2 - s_{12} + s_{23})y_{43} + m_4^2 - m_1^2 + s_{12}, \\ s_{34} &= (m_1^2 - s_{13} + s_{23} - m_2^2)\lambda_3 + (m_1^2 - m_3^2 - s_{13})y_{43} + m_4^2 - m_1^2 + s_{13}. \end{aligned} \quad (4.51)$$

There is an essential difference between functional equation Eq. (4.49) obtained from Eq. (3.12) and functional equation (4.50) derived from Eq. (3.21). For example, at $m_1^2 = m_2^2 = m_3^2 = m_4^2 = 0$, Eq. (4.49) becomes trivial while from Eq. (4.50) for the integral

$$I_3^{(d)}(0, 0, 0, s_{23}, s_{13}, s_{12}) \equiv I_3^{(d)}(s_{23}, s_{13}, s_{12}) \quad (4.52)$$

we obtain nontrivial functional equation:

$$\begin{aligned} &I_3^{(d)}(s_{23}, s_{13}, s_{12}) \\ &= (1 - \lambda_3 - y_{43}) I_3^{(d)}(s_{12}(1 - \lambda_3 - y_{43}) + s_{23}y_{43}, s_{23}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - y_{43})) \\ &+ \lambda_3 I_3^{(d)}(s_{13}y_{43} + \lambda_3 s_{12}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - y_{43}), s_{13}) \\ &+ y_{43} I_3^{(d)}(s_{12}, s_{12}(1 - \lambda_3 - y_{43}) + s_{23}y_{43}, s_{13}y_{43} + \lambda_3 s_{12}), \end{aligned} \quad (4.53)$$

where λ_3 is a root of the quadratic equation

$$s_{12}\lambda_3^2 - (s_{12} - s_{12}y_{43} - s_{13}y_{43} + s_{23}y_{43})\lambda_3 + y_{43}^2s_{13} - y_{43}s_{13} = 0. \quad (4.54)$$

If one argument of $I_3^{(d)}(s_{23}, s_{13}, s_{12})$ is zero then by applying functional equation (4.53) such an integral can be expressed in terms of integrals $I_3^{(d)}$ with two arguments equal to zero. For example, at $s_{23} = 0$ and $y_{43} = s_{12}/(s_{12} - s_{13})$ the relation (4.53) becomes:

$$I_3^{(d)}(0, s_{13}, s_{12}) = \frac{s_{12}}{s_{12} - s_{13}} I_3^{(d)}(s_{12}, 0, 0) - \frac{s_{13}}{s_{12} - s_{13}} I_3^{(d)}(0, 0, s_{13}). \quad (4.55)$$

This is a typical example how functional equations can be used to simplify evaluation of an integral by reducing it to a combination of integrals with fewer number of arguments.

At $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$, similar to the previous case, Eq.(4.49) degenerate while from Eq.(4.50) for the integral

$$I_3^{(d)}(m^2, m^2, m^2, s_{23}, s_{13}, s_{12}) \equiv I_3^{(d)}(m^2; s_{23}, s_{13}, s_{12}) \quad (4.56)$$

we obtain nontrivial functional equation:

$$\begin{aligned} & I_3^{(d)}(m^2; s_{23}, s_{13}, s_{12}) \\ &= (1 - \lambda_3 - y_{43}) I_3^{(d)}(m^2; s_{12}(1 - \lambda_3 - y_{43}) + s_{23}y_{43}, s_{23}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - y_{43})) \\ &+ \lambda_3 I_3^{(d)}(m^2; s_{13}y_{43} + \lambda_3 s_{12}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - y_{43}), s_{13}) \\ &+ y_{43} I_3^{(d)}(m^2; s_{12}, s_{12}(1 - \lambda_3 - y_{43}) + s_{23}y_{43}, s_{13}y_{43} + \lambda_3 s_{12}), \end{aligned} \quad (4.57)$$

where λ_3 is a root of the quadratic equation

$$s_{12}\lambda_3^2 - (s_{12} - s_{12}y_{43} - s_{13}y_{43} + s_{23}y_{43})\lambda_3 + y_{43}^2s_{13} - y_{43}s_{13} = 0. \quad (4.58)$$

Eqs. (4.57), (4.58) are identical to Eqs. (4.53), (4.54) respectively and therefore functional equation for the integral with massless propagators and functional equation for the integral with all masses equal are the same. Eq. (4.57) at $s_{23} = 0$ and $y_{43} = s_{12}/(s_{12} - s_{13})$ leads to the relation similar to (4.55):

$$I_3^{(d)}(m^2; 0, s_{13}, s_{12}) = \frac{s_{12}}{s_{12} - s_{13}} I_3^{(d)}(m^2; s_{12}, 0, 0) - \frac{s_{13}}{s_{12} - s_{13}} I_3^{(d)}(m^2; 0, 0, s_{13}). \quad (4.59)$$

This is not surprising because coefficients of the Eq. (4.57) are mass independent and in the integrand m^2 and $i\epsilon$ appear in the covariant combination $m^2 - i\epsilon$. For this reason the similarity of functional equations for massless integrals and integrals with all masses equal take place for integrals with more external legs and more loops.

4.3 Functional equations for one-loop box type integrals

Functional equations for the box type integrals can be obtained by multiplying relation (3.12) by two propagators, or by multiplying relation (3.21) by one propagator and then integrating over momentum k_1 . Yet another relation can be obtained just by integrating Eq. (3.28) over

momentum k_1 :

$$\begin{aligned}
& I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
&= \lambda_4 I_4^{(d)}(m_5^2, m_2^2, m_3^2, m_4^2; s_{25}, s_{23}, s_{34}, s_{45}, s_{24}, s_{35}) \\
&+ y_{52} I_4^{(d)}(m_1^2, m_5^2, m_3^2, m_4^2; s_{15}, s_{35}, s_{34}, s_{14}, s_{45}, s_{13}) \\
&+ y_{53} I_4^{(d)}(m_1^2, m_2^2, m_5^2, m_4^2; s_{12}, s_{25}, s_{45}, s_{14}, s_{24}, s_{15}) \\
&+ (1 - y_{52} - y_{53} - \lambda_4) I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_5^2; s_{12}, s_{23}, s_{35}, s_{15}, s_{25}, s_{13}). \tag{4.60}
\end{aligned}$$

Here λ_4 is defined in Eq. (3.33) and m_5, y_{52}, y_{53} are arbitrary parameters and

$$\begin{aligned}
s_{15} &= \lambda_4(m_4^2 - s_{14} - m_1^2) + y_{53}(m_4^2 - m_3^2 - s_{14} + s_{13}) + y_{52}(m_4^2 - m_2^2 - s_{14} + s_{12}) \\
&\quad + s_{14} + m_5^2 - m_4^2, \\
s_{25} &= \lambda_4(m_4^2 - m_1^2 - s_{24} + s_{12}) + y_{53}(m_4^2 - m_3^2 - s_{24} + s_{23}) + y_{52}(m_4^2 - m_2^2 - s_{24}) \\
&\quad + s_{24} + m_5^2 - m_4^2, \\
s_{35} &= \lambda_4(m_4^2 - m_1^2 - s_{34} + s_{13}) + y_{53}(m_4^2 - m_3^2 - s_{34}) + y_{52}(m_4^2 - m_2^2 - s_{34} + s_{23}) \\
&\quad + s_{34} + m_5^2 - m_4^2, \\
s_{45} &= \lambda_4(s_{14} + m_4^2 - m_1^2) + y_{53}(s_{34} + m_4^2 - m_3^2) + y_{52}(s_{24} + m_4^2 - m_2^2) \\
&\quad + m_5^2 - m_4^2. \tag{4.61}
\end{aligned}$$

Arbitrary parameters in this functional equation can be chosen from the requirement of simplicity of evaluation of integrals on the right hand - side of Eq. (4.60) or from some other requirements. For example, one can choose these parameters by transforming arguments to a certain kinematical region needed for analytic continuation of the original integral.

4.4 Functional equation for two-loop vertex type integral

The method described in the previous section can be applied to multi loop integrals. Consider, for example, integral corresponding to the diagram given in Figure 3. If we multiply Eq. (3.12)

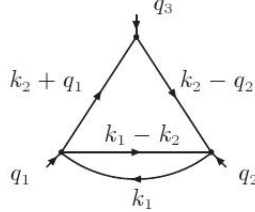


Figure 3: Diagram corresponding to $R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2)$

by the one-loop integral depending on k_1

$$\int \frac{d^d k_2}{[k_2^2 - m_4^2][(k_1 - k_2)^2 - m_5^2]} \tag{4.62}$$

and integrate with respect to momentum k_1 then we obtain functional equation

$$\begin{aligned}
& R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\
&= \alpha R(0, m_2^2, m_3^2, m_4^2; Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\
&+ (1 - \alpha) R(m_1^2, 0, m_3^2, m_4^2; q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2), \tag{4.63}
\end{aligned}$$

where

$$R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) = \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2 + q_1)^2 - m_1^2][(k_2 - q_2)^2 - m_2^2][k_1^2 - m_3^2][(k_1 - k_2)^2 - m_4^2]}, \quad (4.64)$$

$$Q^2 = (q_1^2 - q_2^2 - m_1^2 + m_2^2)\alpha + q_2^2 - m_2^2, \quad (4.65)$$

$$\alpha = \frac{q_3^2 - m_1^2 + m_2^2 \pm \sqrt{\Delta}}{2q_3^2}, \quad (4.66)$$

$$\Delta = q_3^4 + m_1^4 + m_2^4 - 2q_3^2 m_1^2 - 2q_3^2 m_2^2 - 2m_1^2 m_2^2. \quad (4.67)$$

Integrals of this type arise, for example, in calculations of two-loop radiative corrections in the electroweak theory. Instead of the integral $R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2)$ one can consider derivative of R with respect to m_3^2 which is UV finite:

$$\begin{aligned} \frac{\partial R}{\partial m_3^2} &\equiv R_3(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\ &= \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2 + q_1)^2 - m_1^2][(k_2 - q_2)^2 - m_2^2][k_1^2 - m_3^2]^2[(k_1 - k_2)^2 - m_4^2]}. \end{aligned} \quad (4.68)$$

Integral R_3 satisfy the following functional equations:

$$\begin{aligned} R_3(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\ = \alpha R_3(0, m_2^2, m_3^2, m_4^2; Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\ + (1 - \alpha) R_3(m_1^2, 0, m_3^2, m_4^2; q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2). \end{aligned} \quad (4.69)$$

This relation can be used for computing basis integral arising in calculation of two-loop radiative correction to the ortho -positronium lifetime. In particular one of these basis integrals corresponds to kinematics $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$, $q_1^2 = q_2^2 = m^2$, $q_3^2 = 4m^2$. In this case relation (4.69) reads

$$R_3(m^2, m^2, m^2, m^2; m^2, m^2, 4m^2) = R_3(0, m^2, m^2, m^2; 0, m^2, m^2). \quad (4.70)$$

Integral on the right hand-side is in fact propagator type integral with one massless line. Applying recurrence relations given in Ref. [7] this integral can be reduced to simpler integral:

$$\begin{aligned} R_3(0, m^2, m^2, m^2; 0, m^2, m^2) \\ = \frac{1}{(i\pi^{d/2})^2} \int \int \frac{d^d k_1 d^d k_2}{k_1^2(k_2^2 - m^2)[(k_1 - k_2)^2 - m^2]^2[(k_1 + q_1)^2 - m^2]} \\ = \frac{2}{3(d-3)} J_{111}^{(d-2)}(m^2), \end{aligned} \quad (4.71)$$

where

$$J_{111}^{(d)}(q^2) = \frac{1}{(i\pi^{d/2})^2} \int \int \frac{d^d k_1 d^d k_2}{(k_1^2 - m^2)[(k_1 - k_2)^2 - m^2][(k_2 - q)^2 - m^2]}. \quad (4.72)$$

At $q^2 = m^2$, the result for $J_{111}^{(d)}(q^2)$ is known [8]:

$$\begin{aligned} \frac{m^{6-2d}}{8} (d-4)^2 (d-2) (d-3) J_{111}^{(d)}(m^2) \\ = {}_3F_2 \left[\begin{matrix} 1, 3-d, \frac{1}{2} \\ \frac{5-d}{2}, \frac{d}{2} \end{matrix}; 1 \right] + (d-3) {}_3F_2 \left[\begin{matrix} 1, \frac{4-d}{2}, \frac{d-1}{2} \\ \frac{3}{2}, d-1 \end{matrix}; 1 \right], \end{aligned} \quad (4.73)$$

and it can be used for the ε expansion of R and R_3 . As was already mentioned at $q_1^2 = q_2^2 = m^2$, $q_3^2 = 4m^2$ integrals on the right hand - side of Eq.(4.63) correspond to propagator type integrals. Analytic result for R reads

$$(d-3)R(m^2, m^2, m^2, m^2, m^2, 4m^2) = \frac{1}{8m^4(d-4)} \left[2m^2(3d-8)(d-3)J_{111}^{(d)} - 3(d-2)^2 \left(T_1^{(d)} \right)^2 \right], \quad (4.74)$$

where

$$T_1^{(d)} = -\Gamma \left(1 - \frac{d}{2} \right) (m^2)^{(d-2)/2}. \quad (4.75)$$

We checked that several first terms in the $\varepsilon = (4-d)/2$ expansion of R and R_3 are in agreement with results of [9]. The main profit from functional equations for R and R_3 comes from the fact that vertex integrals were expressed in terms of simpler, propagator type integrals.

5 Deriving functional equation by deforming propagators

The method described in the previous section does not work for deriving functional equations for all kinds of Feynman integrals. For example, we did not find functional equation for the two-loop vacuum type integral given in Figure 4.

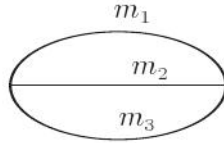


Figure 4: Two-loop vacuum type diagram

In this section we shall describe another method that extends the class of integrals for which we can obtain functional equations. The method is based on transformation of functional equations for some auxiliary integrals depending on arbitrary parameters into functional equations for integrals of interest. Such functional equations will be derived from algebraic relations for ‘deformed propagators’ which will be defined in the next section. These auxiliary integrals will be transformed into α parametric representation. In general characteristic polynomials of these integrals in α parametric representation differ from those for the investigated integral. Functional equation for the integral of interest can be obtained in case when it will be possible to map characteristic polynomials of auxiliary integrals with ‘deformed propagators’ to characteristic polynomials of this integral. Such a mapping will be performed by rescaling α parameters and appropriate choice of arbitrary ‘deforming parameters’.

5.1 Algebraic relations for products of deformed propagators

In the previous section to derive functional equation we added to our consideration a propagator with combination of external momenta taken with arbitrary scalar coefficient. Now we consider

generalization of this method.

To find functional equation for L -loop Feynman integral depending on E - external momenta we start from the relation of the form

$$\prod_{r=1}^n \frac{1}{D_r} = \frac{1}{D_{n+1}} \sum_{r=1}^n x_r \prod_{\substack{j=1 \\ j \neq r}}^n \left(\frac{1}{D_j} \right), \quad (5.76)$$

where D_j is defined as:

$$D_j = Q_j^2 - m_j^2 + i\epsilon, \quad (5.77)$$

with

$$Q_j = \sum_{l=1}^L a_{jl}^{(j)} k_l + \sum_{l=1}^{E+1} b_{jl}^{(j)} p_l, \quad (5.78)$$

and $a_{jl}^{(j)}, b_{jl}^{(j)}$ for the time being are arbitrary scalar parameters. Some of these parameters as well as x_r will be fixed from the equation (5.76). Another part of these parameters will be fixed from the requirement that the product of propagators in (5.76) should correspond to the integrand of the integral with the considered topology. We would like to remark that instead of deformation of propagators proposed in Eqs. (5.77),(5.78) one can use other deformations. For example, all terms in denominators of propagators can be taken with arbitrary scalar coefficients:

$$Q_j^2 = \sum_{l,r=1}^L a_{rl}^{(j)} (k_r \cdot k_l) + \sum_{l=1}^L \sum_{r=1}^{E+1} b_{rl}^{(j)} (p_r \cdot k_l) + \sum_{r,l=1}^{E+1} c_{rl}^{(j)} (p_r \cdot p_l). \quad (5.79)$$

To establish algebraic relation (5.76) we put all terms over a common denominator and then equate coefficients in front scalar products depending on integration momenta. Solving obtained system of equations gives some restrictions on the scalar parameters.

In general integrals obtained by integrating products of ‘deformed propagators’ will not correspond to usual Feynman integrals. Further restrictions on parameters should be imposed in order to obtain relations between integrals corresponding to Feynman integrals coming from a realistic quantum field theory models.

5.2 Functional equation for two-loop vacuum type integral with arbitrary masses

As an example, let us consider derivation of functional equation for the two-loop vacuum type integral given in Figure 4:

$$J_0^{(d)}(m_1^2, m_2^2, m_3^2) = \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2 - m_1^2)((k_1 - k_2)^2 - m_2^2)(k_2^2 - m_3^2)}. \quad (5.80)$$

Analytic expression for this integral was presented in Ref [10]. Instead of this integral we will first consider an auxiliary integral with integrand made from ‘deformed propagators’ defined in Eqs.(5.77), (5.78):

$$\tilde{J}_0^{(d)}(m_1^2, m_2^2, m_3^2) = \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{D_1 D_2 D_3}, \quad (5.81)$$

where

$$\begin{aligned} D_1 &= (a_1 k_1 + a_2 k_2)^2 - m_1^2 + i\epsilon, \quad D_2 = (b_1 k_1 + b_2 k_2)^2 - m_2^2 + i\epsilon, \\ D_3 &= (h_1 k_1 + h_2 k_2)^2 - m_3^2 + i\epsilon. \end{aligned} \quad (5.82)$$

For the product of three deformed propagators one can try to find an algebraic relation of the form:

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_4 D_2 D_3} + \frac{x_2}{D_1 D_4 D_3} + \frac{x_3}{D_1 D_2 D_4}, \quad (5.83)$$

where D_1, D_2, D_3 are defined in Eq.(5.82) and

$$D_4 = (r_1 k_1 + r_2 k_2)^2 - m_4^2. \quad (5.84)$$

Here m_k are arbitrary masses, x_k, a_j, b_i, h_s, r_l are undetermined parameters and k_1, k_2 will be integration momenta.

Putting in Eq. (5.83) all over a common denominator and equating to zero coefficients in front of different products of k_1^2, k_2^2 and $k_1 k_2$ leads to the system of equations:

$$\begin{aligned} r_1^2 - x_1 a_1^2 - x_2 b_1^2 - x_3 h_1^2 &= 0, \quad r_1 r_2 - x_1 a_1 a_2 - x_2 b_1 b_2 - x_3 h_1 h_2 = 0, \\ r_2^2 - x_1 a_2^2 - x_2 b_2^2 - x_3 h_2^2 &= 0, \quad m_4^2 - x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2 = 0. \end{aligned} \quad (5.85)$$

Solving this system for r_1, x_1, x_2, x_3 we have:

$$\begin{aligned} r_1 &= r_2 \lambda, \\ Ax_1 &= r_2^2 (h_1 h_2 m_2^2 - b_2 b_1 m_3^2) + b_2 h_2 (b_1 h_2 - b_2 h_1) m_4^2 - r_2^2 (h_2^2 m_2^2 - m_3^2 b_2^2) \lambda, \\ Ax_2 &= -r_2^2 (m_1^2 h_2 h_1 - a_2 a_1 m_3^2) - a_2 h_2 (a_1 h_2 - a_2 h_1) m_4^2 + r_2^2 (h_2^2 m_1^2 - m_3^2 a_2^2) \lambda, \\ Ax_3 &= r_2^2 (m_1^2 b_2 b_1 - a_2 a_1 m_2^2) + a_2 b_2 (a_1 b_2 - a_2 b_1) m_4^2 - r_2^2 (b_2^2 m_1^2 - m_2^2 a_2^2) \lambda, \end{aligned} \quad (5.86)$$

where λ is a root of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0, \quad (5.87)$$

and

$$\begin{aligned} A &= b_2 h_2 (b_1 h_2 - h_1 b_2) m_1^2 + a_2 h_2 (h_1 a_2 - a_1 h_2) m_2^2 + a_2 b_2 (a_1 b_2 - b_1 a_2) m_3^2, \\ B &= (h_1 b_2 - b_1 h_2) (b_1 h_2 + h_1 b_2) m_1^2 + (a_1 h_2 - h_1 a_2) (a_1 h_2 + h_1 a_2) m_2^2 \\ &\quad + (b_1 a_2 - a_1 b_2) (a_1 b_2 + b_1 a_2) m_3^2, \\ C &= b_1 h_1 (b_1 h_2 - h_1 b_2) m_1^2 + a_1 h_1 (a_2 h_1 - a_1 h_2) m_2^2 + a_1 b_1 (a_1 b_2 - b_1 a_2) m_3^2 \\ &\quad + (b_2 h_1 - b_1 h_2) (a_1 h_2 - a_2 h_1) (a_1 b_2 - a_2 b_1) m_4^2. \end{aligned} \quad (5.88)$$

In order to obtain functional equation for the integral $J_0^{(d)}$, we integrate first both sides of the Eq. (5.83) with respect to k_1, k_2 and then convert these integrals into the α -parametric representation. Transforming all propagators into a parametric form

$$\frac{1}{(k^2 - m^2 + i\epsilon)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^\infty d\alpha \alpha^{\nu-1} \exp [i\alpha(k^2 - m^2 + i\epsilon)], \quad (5.89)$$

and using the d - dimensional Gaussian integration formula

$$\int d^d k \exp [i(ak^2 + 2(pk))] = i \left(\frac{\pi}{ia} \right)^{\frac{d}{2}} \exp \left[-\frac{ip^2}{a} \right], \quad (5.90)$$

we can easily evaluate the integrals over loop momenta. The final result is:

$$\tilde{J}_0^{(d)}(m_1^2, m_2^2, m_3^2) = i^{1-d} \prod_{j=1}^3 \int_0^\infty \dots \int_0^\infty \frac{d\alpha_j}{[\tilde{D}(\alpha)]^{\frac{d}{2}}} \exp \left[-i \sum_{l=1}^3 \alpha_l (m_l^2 - i\epsilon) \right], \quad (5.91)$$

where the polynomial

$$\tilde{D} = (a_1 b_2 - a_2 b_1)^2 \alpha_1 \alpha_2 + (a_1 h_2 - a_2 h_1)^2 \alpha_1 \alpha_3 + (b_1 h_2 - b_2 h_1)^2 \alpha_2 \alpha_3, \quad (5.92)$$

differs from the appropriate D polynomial of the two-loop vacuum integral defined in Eq.(5.80)

$$D = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3. \quad (5.93)$$

Changing in Eq. (5.91) integration variables

$$\alpha_j \rightarrow \alpha_j \theta_j^2 \quad (5.94)$$

with

$$\theta_1 = b_1 h_2 - b_2 h_1, \quad \theta_2 = a_1 h_2 - a_2 h_1, \quad \theta_3 = a_1 b_2 - a_2 b_1, \quad (5.95)$$

leads to the relation:

$$\tilde{D} = (b_1 h_2 - b_2 h_1)^2 (a_1 h_2 - a_2 h_1)^2 (a_1 b_2 - a_2 b_1)^2 (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3). \quad (5.96)$$

Therefore integral $\tilde{J}_0^{(d)}$ with deformed propagators is proportional to $J_0^{(d)}$ with modified arguments:

$$\begin{aligned} \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{D_1 D_2 D_3} &= i^{1-d} [\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{D^{\frac{d}{2}}} \exp[-i\mathcal{M}] \\ &= [\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2) \end{aligned} \quad (5.97)$$

where

$$\mathcal{M} = \alpha_1 \theta_1^2 m_1^2 + \alpha_2 \theta_2^2 m_2^2 + \alpha_3 \theta_3^2 m_3^2. \quad (5.98)$$

With the aid of Eq. (5.97) relation (5.83) integrated with respect to k_1, k_2 can be written as a combination of integrals $J_0^{(d)}$ with different arguments:

$$\begin{aligned} &[\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2) \\ &= x_1 [\theta_1^2 \theta_6^2 \theta_4^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_4^2, \theta_6^2 m_2^2, \theta_4^2 m_3^2) \\ &+ x_2 [\theta_2^2 \theta_5^2 \theta_6^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_6^2 m_1^2, \theta_2^2 m_4^2, \theta_5^2 m_3^2) \\ &+ x_3 [\theta_3^2 \theta_4^2 \theta_5^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_4^2 m_1^2, \theta_5^2 m_2^2, \theta_3^2 m_4^2), \end{aligned} \quad (5.99)$$

where $\theta_1, \theta_2, \theta_3$ are defined in (5.95) and

$$\theta_4 = r_1 b_2 - r_2 b_1, \quad \theta_5 = r_1 a_2 - r_2 a_1, \quad \theta_6 = r_1 h_2 - r_2 h_1. \quad (5.100)$$

Now we consider relation (5.99) as an equation for integrals in momentum representation. By rescaling integration variables k_1, k_2 in the integral on the left hand - side

$$k_1 = (\theta_1 \theta_2 \theta_3)^{\frac{1}{2}} \tilde{k}_1, \quad k_2 = (\theta_1 \theta_2 \theta_3)^{\frac{1}{2}} \tilde{k}_2, \quad (5.101)$$

and performing analogous changes for the integrals on the right hand - side we obtain the relation

$$\begin{aligned}
& \frac{1}{\theta_1\theta_2\theta_3} J_0^{(d)} \left(\frac{\theta_1}{\theta_2\theta_3} m_1^2, \frac{\theta_2}{\theta_1\theta_3} m_2^2, \frac{\theta_3}{\theta_1\theta_2} m_3^2 \right) \\
&= \frac{x_1}{\theta_1\theta_4\theta_6} J_0^{(d)} \left(\frac{\theta_1}{\theta_4\theta_6} m_4^2, \frac{\theta_6}{\theta_1\theta_4} m_2^2, \frac{\theta_4}{\theta_1\theta_6} m_3^2 \right) \\
&+ \frac{x_2}{\theta_2\theta_5\theta_6} J_0^{(d)} \left(\frac{\theta_6}{\theta_2\theta_5} m_1^2, \frac{\theta_2}{\theta_6\theta_5} m_4^2, \frac{\theta_5}{\theta_2\theta_6} m_3^2 \right) \\
&+ \frac{x_3}{\theta_3\theta_4\theta_5} J_0^{(d)} \left(\frac{\theta_4}{\theta_3\theta_5} m_1^2, \frac{\theta_5}{\theta_3\theta_4} m_2^2, \frac{\theta_3}{\theta_5\theta_4} m_4^2 \right). \tag{5.102}
\end{aligned}$$

In terms of redefined masses M_1, M_2, M_3 related to original masses m_1, m_2, m_3 as

$$m_1^2 = \frac{\theta_2\theta_3}{\theta_1} M_1^2, \quad m_2^2 = \frac{\theta_1\theta_3}{\theta_2} M_2^2, \quad m_3^2 = \frac{\theta_1\theta_2}{\theta_3} M_3^2, \tag{5.103}$$

equation (5.102) reads

$$\begin{aligned}
J_0^{(d)}(M_1^2, M_2^2, M_3^2) &= \frac{\theta_2\theta_3}{\theta_4\theta_6} \tilde{x}_1 J_0^{(d)} \left(\frac{\theta_1}{\theta_4\theta_6} m_4^2, \frac{\theta_3\tilde{\theta}_6}{\theta_2\theta_4} M_2^2, \frac{\theta_2\tilde{\theta}_4}{\theta_3\theta_6} M_3^2 \right) \\
&+ \frac{\theta_1\theta_3}{\theta_5\theta_6} \tilde{x}_2 J_0^{(d)} \left(\frac{\theta_3\tilde{\theta}_6}{\theta_1\theta_5} M_1^2, \frac{\theta_2}{\theta_5\theta_6} m_4^2, \frac{\theta_1\tilde{\theta}_5}{\theta_3\theta_6} M_3^2 \right) \\
&+ \frac{\theta_1\theta_2}{\theta_4\theta_5} \tilde{x}_3 J_0^{(d)} \left(\frac{\theta_2\tilde{\theta}_4}{\theta_1\theta_5} M_1^2, \frac{\theta_1\tilde{\theta}_5}{\theta_2\theta_4} M_2^2, \frac{\theta_3}{\theta_4\theta_5} m_4^2 \right), \tag{5.104}
\end{aligned}$$

where $\tilde{x}_i, \tilde{\theta}_j$ are defined from Eqs. (5.86),(5.88) with redefined masses. After simplifications Eq. (5.104) takes a simpler form:

$$\begin{aligned}
& J_0^{(d)}(M_1^2, M_2^2, M_3^2) \\
&= J_0^{(d)}(\rho_1 + \rho_2 + M_1^2 - M_2^2 - M_3^2, \rho_1, \rho_2) \\
&+ J_0^{(d)}(\rho_3, M_3^2 - \rho_2, \rho_3 - \rho_2 - M_1^2 + M_2^2) \\
&- J_0^{(d)}(\rho_3 - M_1^2, \rho_1 - M_2^2, \rho_1 + \rho_3 - M_3^2), \tag{5.105}
\end{aligned}$$

where

$$\begin{aligned}
\rho_1 &= \frac{(M_3^2 + M_2^2 - M_1^2)\theta_1\theta_2\theta_3 - 2b_2h_2\theta_2\theta_3m_4^2 - \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_2^2 - 2b_2^2\theta_2^2m_4^2} M_2^2 \\
\rho_2 &= \frac{(M_3^2 + M_2^2 - M_1^2)\theta_1\theta_2\theta_3 - 2b_2h_2\theta_2\theta_3m_4^2 + \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_3^2 - 2h_2^2\theta_3^2m_4^2} M_3^2 \\
\rho_3 &= \frac{(M_1^2 - M_2^2 + M_3^2)\theta_1\theta_2\theta_3 + 2a_2h_2\theta_1\theta_3m_4^2 + \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_1^2 - 2a_2^2\theta_1^2m_4^2} M_1^2, \tag{5.106}
\end{aligned}$$

with

$$\begin{aligned}
\delta_2 &= \theta_1^2\theta_2^2\theta_3^2\Delta_2 + 4A\theta_1\theta_2\theta_3m_4^2, \\
\Delta_2 &= M_1^4 + M_2^4 + M_3^4 - 2M_1^2M_2^2 - 2M_1^2M_3^2 - 2M_2^2M_3^2. \tag{5.107}
\end{aligned}$$

The coefficients A , B and C can be expressed in terms of M_i , θ_j :

$$\begin{aligned} A &= b_2 h_2 \theta_2 \theta_3 M_1^2 - a_2 h_2 \theta_1 \theta_3 M_2^2 + a_2 b_2 \theta_1 \theta_2 M_3^2, \\ B &= -(b_1 h_2 + b_2 h_1) \theta_2 \theta_3 M_1^2 + (a_1 h_2 + a_2 h_1) \theta_1 \theta_3 M_2^2 - (a_1 b_2 + a_2 b_1) \theta_1 \theta_2 M_3^2, \\ C &= b_1 h_1 \theta_2 \theta_3 M_1^2 - \theta_1 \theta_3 a_1 h_1 M_2^2 + a_1 b_1 \theta_1 \theta_2 M_3^2 - \theta_1 \theta_2 \theta_3 m_4^2. \end{aligned} \quad (5.108)$$

One can easily observe that due to relations

$$\begin{aligned} \rho_1 \rho_2 &= M_2^2 M_3^3, \\ \rho_1 \rho_3 &= M_1^2 \rho_1 + M_2^2 \rho_3, \\ \rho_2 \rho_3 &= M_3^2 \rho_3 - M_1^2 M_3^2, \end{aligned} \quad (5.109)$$

parameters ρ_2 and ρ_3 can be expressed in terms of ρ_1 and therefore all the auxiliary parameters introduced in derivation of the functional equation will be absorbed only in one parameter - ρ_1 .

We would like to notice that Eq. (5.105) is valid for integrals but not for their integrands. This is due to the fact that the factor in front of integral that comes from the scaling of α parameters in parametric integral is not fully compensated by scaling momenta given in Eq. (5.101).

At $m_4 = 0$ the dependence on all parameters a_i, b_j, h_k in Eqs. (5.104), (5.105) drops out and the integral $J_0^{(d)}(M_1^2, M_2^2, M_3^2)$ reduces to a combination of simpler integrals:

$$\begin{aligned} &J_0^{(d)}(M_1^2, M_2^2, M_3^2) \\ &= J_0^{(d)}\left(0, \frac{-M_1^2 + M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 + M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\ &+ J_0^{(d)}\left(\frac{M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, 0, \frac{M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\ &- J_0^{(d)}\left(\frac{-M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}, 0\right). \end{aligned} \quad (5.110)$$

Analytic expression for the integral $J_0^{(d)}$ with one mass equal to zero is known [10]. Under assumption that $|m_3^2 - m_2^2| \leq |m_3^2|$ it reads

$$J_0^{(d)}(0, m_2^2, m_3^2) = \frac{\pi^2(2-d)(m_2^2 m_3^2)^{\frac{d-2}{2}}}{2(d-3)m_3^2 \left(\Gamma\left(\frac{d}{2}\right) \sin \frac{\pi d}{2}\right)^2} {}_2F_1\left[1, 2 - \frac{d}{2}; 4 - d; 1 - \frac{m_2^2}{m_3^2}\right]. \quad (5.111)$$

From functional equation (5.110) as a by-product one can get a new hypergeometric representation for the one-loop massless vertex type integral. In Ref. [11] an interesting relation between the dimensionally regularized one-loop vertex type integral $I_3^{(d)}(m_1^2, m_2^2, m_3^2)$ and the two-dimensional integral $J_0^{(6-d)}(m_1^2, m_2^2, m_3^2)$ was discovered

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = \frac{\Gamma\left(3 - \frac{d}{2}\right)}{\Gamma(d-3)} (-s_{23} s_{13} s_{12})^{\frac{d-4}{2}} J_0^{(6-d)}(s_{23}, s_{13}, s_{12}). \quad (5.112)$$

Functional equation (5.110) with $J_0^{(d)}(0, m_2^2, m_3^2)$ defined in Eq. (5.111) provide us a new hypergeometric representation for the integral $I_3^{(d)}$ with massless propagators. Formula for the one-loop massless vertex integral in terms of other Gauss' hypergeometric functions is given in Ref. [12].

6 Conclusions

Finally, we summarize what we have accomplished in this paper.

First of all, we formulated new methods for deriving functional equations for Feynman integrals. These methods are rather simple and do not use any kind of integration by parts techniques.

Second, it was shown that integrals with many kinematic arguments can be reduced to a combination of simpler integrals with fewer arguments. In our future publications we are going to demonstrate that in some cases applying functional equations one can reduce, the so-called, master integrals to a combination of simpler integrals from, what we would like to call, a ‘universal’ basis of integrals.

The method based on algebraic relations for ‘deformed propagators’ can be used not only for vacuum type of integrals but also for integrals depending on external momenta. In the present paper we considered rather particular cases of functional equations. The systematic investigation and classification of the proposed functional equations requires application of the methods of algebraic geometry and group theory.

At the present moment it is not quite clear whether functional equations derivable from recurrence relations can be reproduced by the methods of algebraic relations between products of propagators described in Section 3 and Section 5.

A detailed consideration of our functional equations and their application to the one-loop integrals with four, five and six external legs as well as to some two- and three- loop Feynman integrals will be presented in future publications.

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